1 Introduction

The first soliton has been observed more than 150 years ago and in the last few years there was a rapidly growing interest in this topic. The expression soliton describes a special form of wave-propagation which means travelling without spreading or breaking up.
Understandably there is a great interest in investigating solitons in optical waveguides. The application of non-spreading pulses seems to be very promising for long-distance transmission while nonlinear superposition laws could be useful in optical data-processing.
Because it is a typical nonlinear effect the theoretical study of solitons is very difficult. However powerful mathematical treatments have been developed and today it is possible to solve some of the fundamental nonlinear equations. The most important step was the *Inverse Scattering Transform* which was first applied by Zakhanov and Shabat to solve the *Nonlinear Schrödinger Equation*. This equation describes the single-mode propagation of a soliton in an optical fibre. In a dimensionless and normalised form it is given by:

$$i\frac{\partial\Psi}{\partial t} = -\frac{m}{2}\frac{\partial^2\Psi}{\partial x^2} + \Psi^*\Psi\Psi \tag{1}$$

The solution of this equation shows some typical soliton properties:

- A sech-profile in space which does not change in time evolution.
- A superposition-law which does not change the shape but only the phase.

Eqn (1) can be derived from a Hamiltonian by application of functional analysis. Therefore note that the canonic conjugate field of Ψ is simply $i\Psi^*$.

$$H = \frac{1}{2}\int [m\Psi_x^*\Psi_x + \Psi^*\Psi^*\Psi\Psi]\,dx \tag{2}$$

For a quantum-mechanical treatment we replace the field-amplitudes $\Psi^*(x)$ and $\Psi(x)$ by operators $\Psi^\dagger(x)$ and $\Psi^\dagger(x)$. They obey the following commutation relations:

$$[\Psi(x), \Psi(y)] = [\Psi^\dagger(x), \Psi^\dagger(y)] = 0 \tag{3}$$

$$[\Psi(x), \Psi(y)] = \delta(x - y) \tag{4}$$

Therefore $\Psi^\dagger(x)$ and $\Psi^\dagger(x)$ are also creation and annihilation operators of photons at these positions. It is possible to solve the quantum-mechanical model by a N-particle *Bethe-Ansatz*.

2 Parametric Amplifier Equations

Another class of equations describes a waveguide with $\chi^{(2)}$-nonlinearity in the presence of dispersion up to second order. The equations in a normalised form are given by:

$$i\frac{\partial \Phi}{\partial t} = -\frac{m_1}{2}\frac{\partial^2 \Phi}{\partial x^2} + \Psi\Phi^* \tag{5}$$

$$i\frac{\partial \Psi}{\partial t} = -\frac{m_2}{2}\frac{\partial^2 \Psi}{\partial x^2} + \frac{1}{2}\Phi^2 + \omega\Psi \tag{6}$$

System (5) was obtained by Raymer and coworkers in [4]. I couples the pump-wave Ψ at frequency 2ω with the subharmonic Φ at ω. The 2ω-term describes phase matching.
The Hamiltonian for the parametric waveguide is given by:

$$H = \frac{1}{2}\int \left\{ m_1\Phi_x^*\Phi_x + m_2\Psi_x^*\Psi_x + \Psi^*\Phi^2 + \Psi\Phi^{*2} + 2\omega\Psi^*\Psi \right\} dx \tag{7}$$

For later use the derivations are shifted by partial integration from Φ and Ψ to the Φ^* and Ψ^* terms. After quantisation the Hamiltonian takes the following form:

$$H = \frac{1}{2}\int \left\{ -m_1\Phi_{xx}^\dagger\Phi - m_2\Psi_{xx}^\dagger\Psi + \Psi^\dagger\Phi^2 + \Psi\Phi^{\dagger 2} + 2\omega\Psi^\dagger\Psi \right\} dx \tag{8}$$

Operators of different fields commute with each other at all positions while the operators for the same field obey the same commutation relations as given in eqns (3-4).
The coupling of the two fields now describes the creation of a photon in the pump-wave and a simultaneous destruction of two photons in the subhamonic-wave and the opposite process.

3 The Bethe-Ansatz

The Bethe-ansatz tries to solve the time independent energy eigenvalue problem with a N-particle ansatz.

$$H|\varphi\rangle = E|\varphi\rangle \tag{9}$$

First we try to solve a two particle problem. The ansatz in its simplest form desribes the possible interactions of one photon in the pump-field and two in the subharmonic:

$$|\varphi\rangle = \int c_1(x_1)\Psi^\dagger(x_1)dx_1|0\rangle + \int\int c_2(x_2,x_3)\Phi^\dagger(x_2)\Phi^\dagger(x_3)dx_2dx_3|0\rangle \tag{10}$$

We insert this ansatz wave-function into eqn (9). We intend to bring anihilation operators to the right, where they act on the ground-state and thereby cancel the whole term. However, some other terms arise because of commutation relations. This has to be done for six terms and for example one of these calculations is given below:

$$\begin{aligned}
&\int\int \Psi(x)\Phi^\dagger(x)\Phi^\dagger(x)\Psi^\dagger(x_1)c_1(x_1)dxdx_1 \\
=&\int\int \Phi^\dagger(x)\Phi^\dagger(x)\{\Psi^\dagger(x_1)\Psi(x) + \delta(x-x_1)\}c_1(x_1)dxdx_1 \\
=&\int\int \Phi^\dagger(x)\Phi^\dagger(x_1)\delta(x-x_1)c_1(x_1)dxdx_1
\end{aligned}$$

In order to equate this tem with the corresponding term of the ansatz we replace x by x_2 and x_1 by x_3. This is possible, because thy are only dummy-variables and have no influence on the value of the integration. Furthermore in the first three terms the second derivatives have to be shifted again. Carrying out similiar procedures for every term we obtain a system of second-order differential equations for the two weighting-functions $c_1(x_1)$ and $c_2(x_2,x_3)$:

$$-\frac{m_2}{2}\frac{\partial^2}{\partial x_1^2}c_1(x_1) + c_2(x_1,x_1) + \omega c_1(x_1) = Ec_1(x_1) \tag{11}$$

$$-\frac{m_1}{2}\left\{\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right\}c_2(x_2,x_3) + \frac{1}{2}c_1(x_2)\delta(x_2-x_3) = Ec_2(x_2,x_3) \tag{12}$$

4 Solution

To solve the system (11) we first introduce a centre of mass and a relative coordinate:

$$\xi = \frac{1}{2}(x_1 + x_2) \tag{13}$$
$$\eta = x_2 - x_3 \tag{14}$$

Therefore the function $c_2(x_2, x_3)$ becomes a function $d_2 = d_2(\xi, \eta)$ and the Laplacian takes the following form:

$$\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{1}{2}\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta^2} \tag{15}$$

The system now becomes:

$$-\frac{m_2}{2}\frac{\partial^2}{\partial x_1^2}c_1(x_1) + d_2(x_1, 0) + \omega c_1(x_1) = Ec_1(x_1) \tag{16}$$
$$-\frac{m_1}{2}\left\{\frac{1}{2}\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \eta^2}\right\} d_2(\xi, \eta) + \frac{1}{2}c_1(\xi)\delta(\eta) = Ed_2(\xi, \eta) \tag{17}$$

Here we replaced the argument of c_1 in eqn (17) which is allowed beause of the δ-function:

$$c_1(x_2)\delta(x_2 - x_3) = c_1(\frac{x_2 + x_3}{2})\delta(x_2 - x_3) = c_1(\xi)\delta(\eta) \tag{18}$$

Next we take care of the coupling δ-function. Therefore we assume a dependence of $|\eta|$ in $c_2(\xi, \eta)$. The change in the derivations is then given by the chain-rule:

$$\begin{aligned}\frac{\partial^2}{\partial \eta^2} &= \frac{\partial}{\partial \eta}\left(\frac{\partial|\eta|}{\partial \eta}\frac{\partial}{\partial|\eta|}\right) \\ &= \frac{\partial}{\partial \eta}\left(\text{sign}(\eta)\frac{\partial}{\partial|\eta|}\right) \\ &= 2\delta(\eta)\frac{\partial}{\partial|\eta|} + \text{sign}^2(\eta)\frac{\partial^2}{\partial|\eta|^2}\end{aligned}$$

The square of the sign-function is, of course, equal to one and therefore eqn (17) becomes:

$$-\frac{m_1}{2}\left\{\frac{1}{2}\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial|\eta|^2}\right\}d_2(\xi,|\eta|)+\left\{\frac{\partial}{|\eta|}d_2(\xi,|\eta|)+\frac{}{2}c_1(\xi)\right\}\delta(\eta)=Ed_2(\xi,|\eta|) \tag{19}$$

If we make the coupling-term, i.e. the second term, vanish, the equation takes a very simple form. Indeed, this can be done. By means of the δ-function the term is zero everywhere except for one point. By setting the term inside the brackets equal to zero at this point we can achieve the simplification on the whole range. This imposes a condition on $c_1(\xi)$.

$$c_1(\xi)=-2\left.\frac{\partial}{\partial|\eta|}d_2(\xi,|\eta|)\right|_{\eta=0} \tag{20}$$

The remaining equation is then given by:

$$-\frac{m_1}{2}\left\{\frac{1}{2}\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial|\eta|^2}\right\}d_2(\xi,|\eta|)=Ed_2(\xi,|\eta|) \tag{21}$$

The solution is very easy, it is only a product of exponentials. Because of (20) the exponent of the $|\eta|$-function must be real to ensure that c_1 is a real function. Therefore the solution decays exponentially with increasing distance between the two coordinates x_2 and x_3. A rising solution would make no physical sense. Hence we find:

$$d_2(\xi,|\eta|)=Ae^{ik_1\xi-c|\eta|} \tag{22}$$

The condition for c_1 then becomes explicitely:

$$c_1(x_1)=Ak_2e^{ik_1x_1} \tag{23}$$

We find that this solves eqn (16) and hence a solution for the whole problem is found. The wave-numbers k_1 and k_2 are related to the energy E and the constants of the Hamiltonian:

$$m_2k_1^2+\frac{2}{c}+2\omega = E \tag{24}$$

$$m_1\left\{\frac{1}{2}k_1^2-2c^2\right\} = E \tag{25}$$

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